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Reformulation of relative approximation operators

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Abstract. Rule induction is an important issue for the application of rough set theory in knowledge discovery in information systems. In this paper, we study the relative rough approximations constructed by combining conditional and decision attributes. For complete information systems, some equivalent descriptions of relative rough approximations are presented. A kind of reformulation of relative approximations is proposed with its basic properties being discussed. In set valued information systems, the relative approximations are induced by certain indiscernibility relation and possible indiscernibility relation. By using the indiscernibility classes induced by certain and possible indiscernibility relations, some basic properties of the relative rough approximations are surveyed. The relationships between the relative rough approximations and the existing rough approximations are investigated.

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1. Introduction

The theory of rough sets was firstly proposed by Pawlak [1]. It is used as an effective tool for data analysis in various fields such as pattern recognition, machine learning, data mining, and so on. The fundamental framework of rough set theory is specified by approximation spaces and approximation operators. Using the notions of upper and lower approximations, knowledge hidden in information systems may be unravelled and expressed in the form of decision rules [2, 3, 4, 5]. Also, some extended rough set models, such as general binary relation based rough set [6], fuzzy rough set [7], covering and fuzzy covering based rough sets [8, 9] variable precision rough set [10], probabilistic rough set [11] were presented to extend the application scope of rough sets.

Pawlak rough approximations are usually derived from equivalence classes induced by indiscernibility relations. It is applicable to complete information systems. Some extensions are imposed on the original rough approximations to deal with incomplete information. Slowinski and Stefanowski introduced the notion of possible indiscernibility between objects [12]. Kryszkiewicz constructed a discernibility relation by giving indiscernibility of a missing value with any value of the attribute [13]. Couso and Dubois express rough approximations by using the degree of possibility that objects belong to the same equivalence class under indiscernibility relations [8]. Nakata and Sakai have examined rough approximations by using possible equivalence classes in the case of information tables containing missing values [14].

Recently, Nakata and Sakai developed the approach using the certain and possible indiscernibility relations in rule induction of incomplete information systems. The approach for dealing with possibilistic information were presented in [15, 16] and some basic properties were surveyed. To dealing with set valued information systems, the rough approximations by combining conditional and decision attributes were developed in [17] which aims at rule induction of the systems. In this paper, we further study the relative rough approximations. By using the indiscernibility classes some reformulations of relative rough approximations are presented with their properties being discussed. From these reformulations, the relationships between the relative rough approximations and the existing rough approximations are investigated.

2. Preliminaries

In this section, we briefly introduce basic notions related to rough sets. We refer to [1] for details.

Definition 2.1 ([1])**.** Let U be a nonempty set and R an equivalence relation on U. (U, R) is called an *approximation space*. For any $X \subseteq U$, the sets

$$\underline{R}(X) = \{x | [x]_R \subseteq X\} \tag{2.1}$$

and

$$\overline{R}(X) = \{x | [x]_R \cap X \neq \varnothing\} \tag{2.2}$$

are called the *lower* and *upper approximations* of X respectively, where $[x]_R$ is the equivalence class with respect to R which containing x.

In rough set theory, the concept X with uncertainty is approximated by its *lower approximation* $\underline{R}(X)$ and *upper approximation* $\overline{R}(X)$. The rough set model has been generalized to general binary relation based rough set [6]. If $R \subseteq U \times U$ is a general binary relation on the universe U, then the lower approximation $\underline{R}(X)$ and upper approximation $\overline{R}(X)$ of X are given by:

$$\underline{R}(X) = \{x | R(x) \subseteq X\} \tag{2.3}$$

$$\overline{R}(X) = \{x | R(x) \cap X \neq \varnothing\} \tag{2.4}$$

where $R(x) = \{y \in U | (x, y) \in R\}$.

The application of rough set theory to knowledge discovery in information systems is one of the core issues of the rough set theory which has been widely studied [3, 4, 18, 19, 20]. An information system S can be described as

$$S = (U, AT, \{D(a)|a \in AT\}),$$

where U is a non-empty finite set of objects called the universe, AT is a non-empty finite set of attributes such that $a : U \to D(a)$ for every $a \in AT$ and $D(a)$ is the domain of attribute a.

Given an information system $S = (U, AT, \{D(a)|a \in AT\})$. For any $A \subseteq AT$, the indiscernibility relation R_A induced by A is defined as follows:

$$R_A = \{(x, y) \in U \times U|\forall a \in A(a(x) = a(y)\} \tag{2.5}$$

If $A = \{a\}$, R_A is also denoted by R_a. By using the indiscernibility relations induced by conditional attributes and decision attributes, the knowledge in decision systems can be expressed as decision rules. For $a, b \in AT$, the positive region [20] $pos_a(b)$ of b related to a is given by $pos_a(b) = \cup_{x\in U} \underline{R_a}([x]_{R_b})$. If $x \in pos_a(b)$, $a(x) = u$ and $b(x) = v$, then x supports a decision rule $a = u \to b = v$ [20].

3. The relative rough approximations in complete information systems

In an information system, when objects are characterized by values of attributes, a set of objects being approximated have some structures. Nakata and Sakai [17] proposed the following relative lower and upper approximations.

Definition 3.1 ([17])**.** Let $S = (U, AT, \{D(a)|a \in AT\})$ be an information system and $a_i, a_j \in AT$, $X \subseteq U$. The *lower approximation* $\underline{apr}_{a_i}(X/a_j)$ and *upper approximation* $\overline{apr}_{a_i}(X/a_j)$ are given by

$$\underline{apr}_{a_i}(X/a_j) = \{x|\exists z \in X\forall y \in U((x, y) \notin R_{a_i} \vee ((y, z) \in R_{a_j} \wedge y \in X))\} \tag{3.1}$$

and

$$\overline{apr}_{a_i}(X/a_j) = \{x|\exists z \in X\exists y \in U((x, y) \in R_{a_i} \wedge (y, z) \in R_{a_j} \wedge y \in X)\} \tag{3.2}$$

By this definition, the approximations are determined by two attributes a_i and a_j and can be used to describing the relationships between equivalence classes induced by R_{a_i} and R_{a_j}. In order to distinguish from the original approximation operators (2.1) and (2.2), the approximation operators $\underline{apr}_{a_i}(X/a_j)$ and $\underline{apr}_{a_i}(X/a_j)$ defined in Definition 3.1 are referred to as relative approximation operators in what follows. Suppose that $a_i(x) = u$ and $a_j(x) = v$. If $x \in \underline{apr}_{a_i}(X/a_j)$, then x consistently supports a decision rule $a_i = u \to a_j = v$ [17]. For the upper approximation, $x \in \overline{apr}_{a_i}(X/a_j)$ implies that x supports an approximate decision rule $a_i = u \to a_j = v$ with accuracy degree $\frac{|[x]_{R_{a_i}} \cap [x]_{R_{a_j}}|}{|[x]_{R_{a_i}}|}$.

Similarly, for $A, B \subseteq AT$ and $X \subseteq U$, The *relative lower approximation* $\underline{apr}_A(X/B)$ and *upper approximation* $\overline{apr}_A(X/B)$ are given by

$$\underline{apr}_A(X/B) = \{x|\exists z \in X\forall y \in U((x, y) \notin R_A \vee ((y, z) \in R_B \wedge y \in X))\} \tag{3.3}$$

$$\overline{apr}_A(X/B) = \{x|\exists z \in X\exists y \in U((x, y) \in R_A \wedge (y, z) \in R_B \wedge y \in X)\} \tag{3.4}$$

The following theorem presents some basic properties of these relative approximation operators.

Theorem 3.2. *Let $S = (U, AT, \{D(a)|a \in AT\})$ be an information system and $a_i, a_j \in AT$, $X \subseteq U$. Then*
(1) $\underline{apr}_{a_i}(X/a_j) = \underline{R_{a_i}}(X) \cap pos_{a_i}(a_j)$,
(2) $\overline{apr}_{a_i}(X/a_j) = \overline{R_{a_i}}(X)$.

Proof. (1) We note that the positive region $pos_{a_i}(a_j) = \cup_{x\in U}\underline{R_{a_i}}([x]_{R_{a_j}})$. Assume that $x \in \underline{apr}_{a_i}(X/a_j)$. Then it follows that there exists $z \in X$ such that for every $y \in U$, we have $(x, y) \in R_{a_i}$ implies that $(y, z) \in R_{a_j}$ and $y \in X$. For any $u \in [x]_{R_{a_i}}$, by $(x, u) \in R_{a_i}$ we have $(u, z) \in R_{a_j}$ and $u \in X$. Thus $[x]_{R_{a_i}} \subseteq X$. So $x \in \underline{R_{a_i}}(X)$. Furthermore, we have $u \in [z]_{R_{a_j}}$. Hence $[x]_{R_{a_i}} \subseteq [z]_{R_{a_j}}$. In other words, $x \in \underline{R_{a_i}}([z]_{R_{a_j}}) \subseteq pos_{a_i}(a_j)$. Therefore we have $\underline{apr}_{a_i}(X/a_j) \subseteq \underline{R_{a_i}}(X) \cap pos_{a_i}(a_j)$.

Conversely, assume that $x \in \underline{R_{a_i}}(X) \cap pos_{a_i}(a_j)$. By $x \in \underline{R_{a_i}}(X)$, we have $[x]_{R_{a_i}} \subseteq X$. By $x \in pos_{a_i}(a_j)$, it follows that there exists $z \in X$ such that $[x]_{R_{a_i}} \subseteq [z]_{R_{a_j}}$. Consequently, for any $y \in U$ with $(x, y) \in R_{a_i}$, we have $y \in [x]_{R_{a_i}}$ and hence $y \in X$ and $(y, z) \in R_{a_j}$. Then $x \in \underline{apr}_{a_i}(X/a_j)$. Thus $\underline{R_{a_i}}(X) \cap pos_{a_i}(a_j) \subseteq \underline{apr}_{a_i}(X/a_j)$.

(2) Assume that $x \in \overline{apr}_{a_i}(X/a_j)$. It follows that there exist $z \in X$ and $y \in U$ such that $(x, y) \in R_{a_i}$, $(y, z) \in R_{a_j}$ and $y \in X$. It is conclude that $y \in [x]_{R_{a_i}}$, $y \in [z]_{R_{a_j}}$, $y \in X$. Then $[x]_{R_{a_i}} \cap [z]_{R_{a_j}} \cap X \neq \varnothing$. By $[x]_{R_{a_i}} \cap X \neq \varnothing$, it follows that $x \in \overline{R_{a_i}}(X)$.

Assume that $x \in \overline{R_{a_i}}(X)$. It follows that $[x]_{R_{a_i}} \cap X \neq \varnothing$. Then there exists $y \in X$ such that $y \in [x]_{R_{a_i}}$. It is conclude that $(x, y) \in R_{a_i}$, $y \in X$ and $(y, y) \in R_{a_j}$. Thus $x \in \overline{apr}_{a_i}(X/a_j)$ as required. □

By this theorem, the relative upper approximation $\overline{apr}_{a_i}(X/a_j)$ is completely determined by attribute a_i, and it has nothing to do with attribute a_j.

Corollary 3.3. *Let $S = (U, AT, \{D(a)|a \in AT\})$ be an information system and $a_i, a_j \in AT$, $y \in U$ and $X \subseteq U$. Then*
(1) $\underline{apr}_{a_i}(X/a_j) = \cup_{x\in X}\underline{R_{a_i}}(X \cap [x]_{R_{a_j}})$,
(2) $\underline{apr}_{a_i}([y]_{R_{a_j}}/a_j) = \underline{R_{a_i}}([y]_{R_{a_j}})$,
(3) $\underline{apr}_{a_i}(X/a_j) \subseteq \underline{R_{a_i}}(X) \subseteq X \subseteq \overline{R_{a_i}}(X) = \overline{apr}_{a_i}(X/a_j)$.

Proof. (1) Assume that $x \in \underline{apr}_{a_i}(X/a_j)$. It follows by Theorem 3.2 (1) that $[x]_{R_{a_i}} \subseteq X$ and $[x]_{R_{a_i}} \subseteq [x]_{R_{a_j}}$. Then we have $[x]_{R_{a_i}} \subseteq X \cap [x]_{R_{a_j}}$ and $x \in \underline{R_{a_i}}(X \cap [x]_{R_{a_j}})$. It is conclude that $\underline{apr}_{a_i}(X/a_j) \subseteq \cup_{x\in X}\underline{R_{a_i}}(X \cap [x]_{R_{a_j}})$.

Conversely, assume that $y \in \cup_{x\in X}\underline{R_{a_i}}(X \cap [x]_{R_{a_j}})$. It follows that there exists $x \in X$ such that $y \in \underline{R_{a_i}}(X \cap [x]_{R_{a_j}})$. Then $[y]_{R_{a_i}} \subseteq X \cap [x]_{R_{a_j}}$. By $[y]_{R_{a_i}} \subseteq X$, we have $y \in \underline{R_{a_i}}(X)$. From $[y]_{R_{a_i}} \subseteq [x]_{R_{a_j}}$, it is conclude that $y \in [x]_{R_{a_j}}$. Thus $[x]_{R_{a_j}} = [y]_{R_{a_j}}$. By Theorem 3.2 (1), $y \in \underline{apr}_{a_i}(X/a_j)$. So $\cup_{x\in X}\underline{R_{a_i}}(X \cap [x]_{R_{a_j}}) \subseteq \underline{apr}_{a_i}(X/a_j)$.

(2) Assume that $x \in \underline{R_{a_i}}([y]_{R_{a_j}})$. It follows that $[x]_{R_{a_i}} \subseteq [y]_{R_{a_j}}$. Then $x \in [y]_{R_{a_j}}$. Thus $[x]_{R_{a_j}} = [y]_{R_{a_j}}$ and $[x]_{R_{a_i}} \subseteq [x]_{R_{a_j}}$. It is conclude that $\underline{R_{a_i}}([y]_{R_{a_j}}) \subseteq \{x|[x]_{R_{a_i}} \subseteq [x]_{R_{a_j}}\}$. So $\underline{apr}_{a_i}([y]_{R_{a_j}}/a_j) = \underline{R_{a_i}}([y]_{R_{a_j}})$ by Theorem 3.2 (1).

(3) By Theorem 3.2, we have $\underline{apr}_{a_i}(X/a_j) \subseteq \underline{R_{a_i}}(X)$ and $\overline{apr}_{a_i}(X/a_j) = \overline{R_{a_i}}(X)$. $\underline{R_{a_i}}(X) \subseteq X \subseteq \overline{R_{a_i}}(X)$ follows from the reflexivity of R_{a_i}. □

Corollary 3.4. *Let $S = (U, AT, \{D(a)|a \in AT\})$ be an information system, $a_i, a_j \in AT$ and $X, Y \subseteq U$. Then*

(1) $\underline{apr}_{a_i}(X \cap Y/a_j) = \underline{apr}_{a_i}(X/a_j) \cap \underline{apr}_{a_i}(Y/a_j)$,
(2) $\overline{apr}_{a_i}(X \cup Y/a_j) = \overline{apr}_{a_i}(X/a_j) \cup \overline{apr}_{a_i}(Y/a_j)$,
(3) $\underline{apr}_{a_i}(X/a_j) \subseteq (\overline{apr}_{a_i}(X^c/a_j))^c$, $\overline{apr}_{a_i}(X/a_j) \subseteq (\underline{apr}_{a_i}(X^c/a_j))^c$, *where $X^c = U - X$ is the complement of X in U.*

Corollary 3.5. *Let $S = (U, AT, \{D(a)|a \in AT\})$ be an information system and $A, B \subseteq AT$ and $X \subseteq U$. Then*

(1) $\underline{apr}_A(X/B) = \underline{R_A}(X) \cap \{x|[x]_{R_A} \subseteq [x]_{R_B}\}$,
(2) $\overline{apr}_A(X/B) = \overline{R_A}(X)$,
(3) $\underline{apr}_A(X/B) \subseteq \underline{R_A}(X) \subseteq X \subseteq \overline{R_A}(X) = \overline{apr}_A(X/B)$.

4. The relative rough approximations in set valued information systems

We consider incomplete information systems $S = (U, AT, \{D(a)|a \in AT\})$, where $a : U \to P(D(a))$ for every $a \in AT$ and $P(D(a))$ is the power set of $D(a)$. $v \in a(x)$ is a possible value that may be the actual one as the value of attribute a in object x. The possible value is the actual one if $|a(x)| = 1$. Sometimes S is also called a set valued information system.

The indiscernibility relation for a in an incomplete information system is expressed by using two relations CR_a and PR_a. CR_a is a certain indiscernibility relation and PR_a is a possible one [17]:

$$CR_a = \{(x, y) \in U \times U | x = y \vee (a(x) = a(y) \wedge |a(x)| = 1)\}$$
$$PR_a = \{(x, y) \in U \times U | x = y \vee a(x) \cap a(y) \neq \varnothing\}$$

The certain indiscernibility relation is an equivalence relation. The possible indiscernibility relation is reflexive and symmetric, but not transitive in general. In addition, it is clear that $CR_a \subseteq PR_a$. In what follows, we denote $PR_a(x) = \{y|(x, y) \in PR_a\}$ and $CR_a(x) = \{y|(x, y) \in CR_a\}$.

By using certain and possible indiscernibility relations, we can obtain four approximation operators in set valued information systems. Certain lower approximation $C\underline{apr}_{a_i}(X/a_j)$ and possible lower approximation $P\underline{apr}_{a_i}(X/a_j)$ are given by [17]:

$$C\underline{apr}_{a_i}(X/a_j) = \{x | \exists z \in X \forall y \in U((x, y) \notin PR_{a_i} \vee ((y, z) \in CR_{a_j} \wedge y \in X))\}$$

$$P\underline{apr}_{a_i}(X/a_j) = \{x | \exists z \in X \forall y \in U((x, y) \notin CR_{a_i} \vee ((y, z) \in PR_{a_j} \wedge y \in X))\}$$

Similarly, certain upper approximation $C\overline{apr}_{a_i}(X/a_j)$ and possible upper approximation $P\overline{apr}_{a_i}(X/a_j)$ are given by:

$$C\overline{apr}_{a_i}(X/a_j) = \{x | \exists z \in X \exists y \in U((x, y) \in CR_{a_i} \wedge (y, z) \in CR_{a_j} \wedge y \in X)\}$$

$P\overline{apr}_{a_i}(X/a_j) = \{x|\exists z \in X \exists y \in U((x,y) \in PR_{a_i} \wedge (y,z) \in PR_{a_j} \wedge y \in X)\}$

We discuss the properties of these rough approximation operators.

Theorem 4.1. *Let $S = (U, AT, \{D(a)|a \in AT\})$ be a set valued information system and $a_i, a_j \in AT$, $X \subseteq U$. Then*
(1) $C\underline{apr}_{a_i}(X/a_j) = \underline{PR_{a_i}}(X) \cap \{x|PR_{a_i}(x) \subseteq CR_{a_j}(x)\}$,
(2) $P\underline{apr}_{a_i}(X/a_j) = \underline{CR_{a_i}}(X) \cap \{x|\exists z \in U(CR_{a_i}(x) \subseteq PR_{a_j}(z))\}$.

Proof. (1) Assume that $x \in C\underline{apr}_{a_i}(X/a_j)$. It follows that there exists $z \in X$ such that for every $y \in U$, we have $(x,y) \in PR_{a_i}$ implies that $(y,z) \in CR_{a_j}$ and $y \in X$. That is to say, $y \in PR_{a_i}(x)$ implies that $y \in CR_{a_j}(z) \cap X$. Then $PR_{a_i}(x) \subseteq CR_{a_j}(z) \cap X$. By $PR_{a_i}(x) \subseteq X$, it is conclude that $x \in \underline{PR_{a_i}}(X)$. By $x \in PR_{a_i}(x)$ and $PR_{a_i}(x) \subseteq CR_{a_j}(z)$, we have $x \in CR_{a_j}(z)$. Thus $CR_{a_j}(z) = CR_{a_j}(x)$ because CR_{a_j} is an equivalence relation. So we have

$$C\underline{apr}_{a_i}(X/a_j) \subseteq \underline{PR_{a_i}}(X) \cap \{x|PR_{a_i}(x) \subseteq CR_{a_j}(x)\}.$$

Conversely, assume that $x \in \underline{PR_{a_i}}(X) \cap \{x|PR_{a_i}(x) \subseteq CR_{a_j}(x)\}$. By $x \in \underline{PR_{a_i}}(X)$, we have $PR_{a_i}(x) \subseteq X$. Then $PR_{a_i}(x) \subseteq CR_{a_j}(x) \cap X$. For every $y \in U$, $y \in PR_{a_i}(x)$ implies that $y \in CR_{a_j}(x)$ and $y \in X$. In other words, $(x,y) \in PR_{a_i}$ implies that $(y,x) \in CR_{a_j}$ and $y \in X$. Thus there exists $z = x \in U$ such that for every $y \in U$ we have $(x,y) \in PR_{a_i}$ implies that $(y,z) = (y,x) \in CR_{a_j}$ and $y \in X$. It follows that $x \in C\underline{apr}_{a_i}(X/a_j)$. So we have

$$\underline{PR_{a_i}}(X) \cap \{x|PR_{a_i}(x) \subseteq CR_{a_j}(x)\} \subseteq C\underline{apr}_{a_i}(X/a_j).$$

(2) Assume that $x \in P\underline{apr}_{a_i}(X/a_j)$. It follows that there exists $z \in X$ such that for every $y \in U$, $(x,y) \in CR_{a_i}$ implies that $(y,z) \in PR_{a_j}$ and $y \in X$. In other words, $y \in CR_{a_i}(x)$ implies that $y \in PR_{a_j}(z) \cap X$. Then $CR_{a_i}(x) \subseteq PR_{a_j}(z) \cap X$. By $CR_{a_i}(x) \subseteq X$, it follows that $x \in \underline{CR_{a_i}}(X)$. Thus we get

$$x \in \underline{CR_{a_i}}(X) \cap \{x|\exists z \in U(CR_{a_i}(x) \subseteq PR_{a_j}(z))\}.$$

Conversely, assume that $x \in \underline{CR_{a_i}}(X) \cap \{x|\exists z \in U(CR_{a_i}(x) \subseteq PR_{a_j}(z))\}$. It follows that $CR_{a_i}(x) \subseteq X$ and there exists hence $z \in U$ such that $CR_{a_i}(x) \subseteq PR_{a_j}(z)$. Then we have $CR_{a_i}(x) \subseteq PR_{a_j}(x) \cap X$. Thus for every $y \in U$, $y \in CR_{a_i}(x)$ implies that $y \in PR_{a_j}(x)$ and $y \in X$. In other words, $(x,y) \in CR_{a_i}$ implies that $(x,y) \in PR_{a_j}$ and $y \in X$. Consequently, we have $(y,x) \in PR_{a_j}$ because PR_{a_j} is a binary symmetric relation on U. It follows that there exists $z = x \in U$ such that for every $y \in U$ we have $(x,y) \in CR_{a_i}$ implies that $(y,z) = (y,x) \in PR_{a_j}$ and $y \in X$. It follows that $x \in P\underline{apr}_{a_i}(X/a_j)$. So $P\underline{apr}_{a_i}(X/a_j) \supseteq \underline{CR_{a_i}}(X) \cap \{x|\exists z \in U(CR_{a_i}(x) \subseteq PR_{a_j}(z))\}$ as required. □

Theorem 4.2. *Let $S = (U, AT, \{D(a)|a \in AT\})$ be a set valued information system and $a_i, a_j \in AT$, $X \subseteq U$. Then*
(1) $C\overline{apr}_{a_i}(X/a_j) = \overline{CR_{a_i}}(X)$,
(2) $P\overline{apr}_{a_i}(X/a_j) = \overline{PR_{a_i}}(X)$.

Proof. (1) Assume that $x \in C\overline{apr}_{a_i}(X/a_j)$. It follows that there exist $z \in X$ and $y \in U$ such that $(x,y) \in CR_{a_i}$, $(y,z) \in CR_{a_j}$ and $y \in X$. It is conclude that $y \in CR_{a_i}(x)$, $y \in CR_{a_j}(z)$, $y \in X$. Then $CR_{a_i}(x) \cap CR_{a_j}(z) \cap X \neq \varnothing$. By $CR_{a_i}(x) \cap X \neq \varnothing$, it follows that $x \in \overline{CR_{a_i}}(X)$.

Assume that $x \in \overline{CR_{a_i}}(X)$. It follows that $CR_{a_i}(x) \cap X \neq \varnothing$. Then there exists $y \in X$ such that $y \in CR_{a_i}(x)$. It is conclude that $(x,y) \in CR_{a_i}$, $y \in X$. Thus there exist $z = y \in X$ such that $(x,y) \in CR_{a_i}$, $y \in X$ and $(y,z) = (y,y) \in CR_{a_j}$ because CR_{a_i} is reflexive. So $x \in C\overline{apr}_{a_i}(X/a_j)$ as required.

(2) We can be proved similarly. □

We note that the upper approximations are determined completely by attribute a_i. By Theorem 4.1 and Theorem 4.2, we have the following properties of and relationships among several approximations.

Corollary 4.3. *Let $S = (U, AT, \{D(a)|a \in AT\})$ be a set valued information system and $a_i, a_j \in AT$ and $X \subseteq U$. Then*

(1) $C\underline{apr}_{a_i}(X/a_j) \subseteq \underline{PR_{a_i}}(X) \subseteq \underline{CR_{a_i}}(X) \subseteq X$,

(2) $C\underline{apr}_{a_i}(X/a_j) \subseteq P\underline{apr}_{a_i}(X/a_j) \subseteq \underline{CR_{a_i}}(X) \subseteq X$,

(3) $X \subseteq C\overline{apr}_{a_i}(X/a_j) = \overline{CR_{a_i}}(X) \subseteq P\overline{apr}_{a_i}(X/a_j) = \overline{PR_{a_i}}(X)$.

Example 4.4. We consider the set valued information system given in Table 1, where $U = \{o_1, o_2, o_3, o_4, o_5, o_6\}$, $AT = \{a_1, a_2\}$. Then the certain and possible indiscernibility relations for attributes are as follows:

Table 1. A decision formal context

	o_1	o_2	o_3	o_4	o_5	o_6
a_1	{x}	{x, y}	{y}	{y}	{w}	{w, z}
a_2	{a,c}	{a,b}	{b}	{b}	{c}	{c}

$CR_{a_1} = \{(o_1,o_1),(o_2,o_2),(o_3,o_3),(o_3,o_4),(o_4,o_3),(o_4,o_4),(o_5,o_5),(o_6,o_6)\}$,

$PR_{a_1} = \{(o_1,o_1),(o_1,o_2),(o_2,o_1),(o_2,o_2),(o_2,o_3),(o_2,o_4),(o_3,o_2),(o_3,o_3),$
$(o_3,o_4),(o_4,o_2),(o_4,o_3),(o_4,o_3),(o_5,o_5),(o_5,o_6),(o_6,o_5),(o_6,o_6)\}$.

Thus we have $CR_{a_1}(o_1) = \{o_1\}$, $CR_{a_1}(o_2) = \{o_2\}$, $CR_{a_1}(o_3) = CR_{a_1}(o_4) = \{o_3, o_4\}$, $CR_{a_1}(o_5) = \{o_5\}$, $CR_{a_1}(o_6) = \{o_6\}$, $PR_{a_1}(o_1) = \{o_1, o_2\}$, $PR_{a_1}(o_2) = \{o_1, o_2, o_3, o_4\}$, $PR_{a_1}(o_3) = PR_{a_1}(o_4) = \{o_2, o_3, o_4\}$, $PR_{a_1}(o_5) = PR_{a_1}(o_6) = \{o_5, o_6\}$. Similarly, for attribute a_2, we have $CR_{a_2}(o_1) = \{o_1\}$, $CR_{a_2}(o_2) = \{o_2\}$, $CR_{a_2}(o_3) = CR_{a_2}(o_4) = \{o_3, o_4\}$, $CR_{a_2}(o_5) = CR_{a_2}(o_6)\{o_5, o_6\}$, $PR_{a_2}(o_1) = \{o_1, o_2, o_5, o_6\}$, $PR_{a_2}(o_2) = \{o_1, o_2, o_3, o_4\}$, $PR_{a_2}(o_3) = PR_{a_2}(o_4) = \{o_2, o_3, o_4\}$, $PR_{a_2}(o_5) = PR_{a_2}(o_6) = \{o_1, o_5, o_6\}$.

Let $X = \{o_2, o_3, o_4\}$. We have $\underline{PR_{a_1}}(X) = \{o_3, o_4\}$, $\{x|PR_{a_1}(x) \subseteq CR_{a_2}(x)\} = \{o_5, o_6\}$ and hence $C\underline{apr}_{a_1}(X/a_2) = \{o_3, o_4\} \cap \{o_5, o_6\} = \varnothing$ by Theorem 4.1. In addition, we have $\underline{CR_{a_1}}(X) = \{o_2, o_3, o_4\}$, and $\{x|\exists z \in U(CR_{a_1}(x) \subseteq PR_{a_2}(z))\} = \{o_1, o_2, o_3, o_4, o_5, o_6\} = U$. By Theorem 4.1, it is conclude that $P\underline{apr}_{a_1}(X/a_2) = \{o_2, o_3, o_4\}$.

Similarly, we have $C\overline{apr}_{a_1}(X/a_2) = \overline{CR_{a_1}}(X) = \{o_2, o_3, o_4\}$, $P\overline{apr}_{a_1}(X/a_2) = \overline{PR_{a_1}}(X) = \{o_1, o_2, o_3, o_4\}$.

5. Concluding remarks

Rough approximation and rule induction are important issues for knowledge discovery in information systems. This paper is devoted to the discussion of relative rough approximations. For complete information systems, a kind of reformulation and some equivalent descriptions of relative approximations are presented. Some basic properties of the relative rough approximations are surveyed. For set valued information systems, the relative approximations are induced by certain indiscernibility relation and possible indiscernibility relation. By using the indiscernibility classes induced by certain and possible indiscernibility relations, some basic properties of the relative rough approximations are surveyed. The relationships between the relative rough approximations and the existing rough approximations are investigated. In further study, the relative rough approximations for possibilistic information systems will be studied.

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